Mathematical Methods in Physics HW6

1. Consider the matrix $A_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Now consider a perturbation given by

$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. For the new matrix, determine the eigenvalues to first order in ϵ and the appropriate zeroth order eigenvectors for corrections in ϵ using degenerate perturbation theory.

Let's get the starting point: $det[A_0 - \lambda I] = 0 = (1-\lambda)(2-\lambda)(1-\lambda) - (2-\lambda) = (2-\lambda)(2-\lambda)\lambda$

$\lambda_1^{(0)} = 0, \ \lambda_{2,1}^{(0)} = 2, \ \lambda_{2,2}^{(0)} = 2$

For $\lambda_1^{(0)} = 0 \quad \Rightarrow \quad A_0 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+c \\ 2b \\ a+c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \hat{x}_1^{(0)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

For $\lambda_{2,i}^{(0)} = 2 \quad \Rightarrow \quad A_0 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+c \\ 2b \\ a+c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix} \quad \Rightarrow \quad \hat{x}_{2,1}^{(0)} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}, \quad \hat{x}_{2,2}^{(0)} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$

For $\lambda_1^{(0)}$: $\quad \lambda_1^{(1)} = \left(\hat{x}_1^{(0)}, A_1 \hat{x}_1^{(0)}\right) = 0$

For $\lambda_{2,i}^{(0)}$: $\quad M_{jk} = \left(\hat{x}_{2,j}^{(0)}, A_1 \hat{x}_{2,k}^{(0)}\right) \quad \Rightarrow \quad M = \begin{pmatrix} (\hat{x}_{2,1}^{(0)}, A_1 \hat{x}_{2,1}^{(0)}) & (\hat{x}_{2,1}^{(0)}, A_1 \hat{x}_{2,2}^{(0)}) \\ (\hat{x}_{2,2}^{(0)}, A_1 \hat{x}_{2,1}^{(0)}) & (\hat{x}_{2,2}^{(0)}, A_1 \hat{x}_{2,2}^{(0)}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$det\left[M - \lambda_{2,i}^{(1)} I\right] = 0 = \left(\frac{1}{2} - \lambda_{2,i}^{(1)}\right)^2 - \frac{1}{4} = \lambda_{2,i}^{(1)^2} - \lambda_{2,i}^{(1)} = \lambda_{2,i}^{(1)}\left(\lambda_{2,i}^{(1)} - 1\right) \quad \Rightarrow \quad \lambda_{2,1}^{(1)} = 0, \ \lambda_{2,2}^{(1)} = 1$

Therefore: $\lambda_1 = 0, \ \lambda_2 = 2, \ \lambda_3 = 2 + \epsilon$

$\lambda_{2,1}^{(1)}$: $M \begin{pmatrix} a \\ b \end{pmatrix} = \lambda_{2,1}^{(1)} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{matrix} \frac{1}{2}a - \frac{1}{2}b = 0 \\ -\frac{1}{2}a + \frac{1}{2}b = 0 \end{matrix} \quad \Rightarrow \quad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = a_{1\,k}$

$\lambda_{2,1}^{(1)}$: $M \begin{pmatrix} a \\ b \end{pmatrix} = \lambda_{2,2}^{(1)} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{matrix} \frac{1}{2}a - \frac{1}{2}b = a \\ -\frac{1}{2}a + \frac{1}{2}b = b \end{matrix} \quad \Rightarrow \quad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = a_{2\,k}$

$y_{2,1}^{(0)} = \sum_k a_{1\,k}\, \hat{x}_{2,k}^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$y_{2,2}^{(0)} = \sum_k a_{2\,k}\, \hat{x}_{2,k}^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

2. Check that your results from problem (1) for the eigenvalues agree with what you would obtain by directly determining the eigenvalues as an expansion in ϵ.

$$det[A-\lambda I] = 0 = det\begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 2+\epsilon-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2(2+\epsilon-\lambda) - (2+\epsilon-\lambda)$$

$= (2+\epsilon-\lambda)[(1-\lambda)^2 - 1] = (2+\epsilon-\lambda)\lambda(\lambda-2) \quad \Rightarrow \quad \lambda_1 = 0,\ \lambda_2 = 2,\ \lambda_3 = 2+\epsilon$ As before!

3. Consider the function $f(x) = \begin{cases} 1 & \text{for irrational } x \\ 0 & \text{for rational } x \end{cases}$ on the closed domain $x \in [0,1]$ with inner product $(f_1, f_2) = \int_0^1 f_1^* f_2 dx$. Find $\|f\|^2$ using both the Riemannian measure as well as the Lebesgue measure. Do they agree?

Well first of all note that $f(x) = -D(x) + 1$ where D is the Dirichlet function discussed in class. To evaluate $\|f\|^2$ we use:

$$\|f\|^2 = (f,f) = \int_0^1 f^* f dx = \int_0^1 (-D+1)(-D+1)dx = \int_0^1 D^2 dx + \int_0^1 dx - 2\int_0^1 D dx$$

Now $\int_0^1 dx = 1$ and we can argue that $\int_0^1 D^2 dx = \int_0^1 D dx$ since:

$$D(x) = \begin{cases} 1 & \text{for rational } x \\ 0 & \text{for irrational } x \end{cases} \quad \Rightarrow D^2(x) = \begin{cases} 1 & \text{for rational } x \\ 0 & \text{for irrational } x \end{cases}$$

Hence $\|f\|^2 = 1 - \int_0^1 D dx$

Furthermore, for the Riemannian measure we obtain either $\int_0^1 D dx = 0$ or $\int_0^1 D dx = 1$ depending on whether we use the max or min of the rectangles to be summed, so this is not well-defined for the Riemannian case. For the Lebesgue measure, we know that $\int_0^1 D dx = 0$, therefore $\|f\|^2 = 1$.

4. a) For the example in class of a space which is not complete, i.e. continuous functions with norm defined by $\|x\| = \int_0^1 |x(t)| dt$, show that for the given sequence $x_n(t) = \begin{cases} 0 & 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ 1 + nt - \frac{n}{2} & \frac{1}{2} - \frac{1}{n} \le t \le \frac{1}{2} \\ 1 & \frac{1}{2} \le t \le 1 \end{cases}$,

$\|x_n - x_m\| = \frac{1}{2}\left|\frac{1}{n} - \frac{1}{m}\right|$, and hence is Cauchy.

To start: $\|x_n - x_m\| = \int_0^1 |x_n(t) - x_m(t)| dt$

If we choose $n < m$ then we know that $x_n(t) > x_m(t)$ for all t, so we can ignore the absolute value.

$$\int_0^1 |x_n(t) - x_m(t)| dt = \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} \left[1 + nt - \frac{n}{2}\right] dt - \int_{\frac{1}{2}-\frac{1}{m}}^{\frac{1}{2}} \left[1 + mt - \frac{m}{2}\right] dt$$

$$= \left[t + \frac{1}{2}nt^2 - \frac{n}{2}t\right]_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} - \left[t + \frac{1}{2}mt^2 - \frac{m}{2}t\right]_{\frac{1}{2}-\frac{1}{m}}^{\frac{1}{2}}$$

$= \left|\frac{1}{2}+\frac{1}{8}n-\frac{n}{4}-\frac{1}{2}+\frac{1}{n}-\frac{1}{8}n-\frac{1}{2n}+\frac{1}{2}+\frac{n}{4}-\frac{1}{2}-\left[\frac{1}{2}+\frac{1}{8}m-\frac{m}{4}-\frac{1}{2}+\frac{1}{m}-\frac{1}{8}m-\frac{1}{2m}+\frac{1}{2}+\frac{m}{4}-\frac{1}{2}\right]\right|$

$= \frac{1}{2}\left(\frac{1}{n}-\frac{1}{m}\right)$ and now we can restore the arbitrariness of which is bigger by adding back in the absolute value giving $\frac{1}{2}\left|\frac{1}{n}-\frac{1}{m}\right|$.

To be Cauchy, we need that for every $\epsilon > 0$, there exists a number $N(\epsilon)$ such that $\|x_n - x_m\| < \epsilon$ for every $n, m > N(\epsilon)$.

In this case we need $\|x_n - x_m\| = \frac{1}{2}\left|\frac{1}{n}-\frac{1}{m}\right| < \epsilon$. Considering all values of n and m, the largest values that this expression may obtain for a given value of either index is when the other is ∞.

Thus we can require $\|x_n - x_\infty\| = \frac{1}{2}\left|\frac{1}{n}-0\right| = \frac{1}{2n} < \epsilon \quad \Rightarrow \quad n > \frac{1}{2\epsilon} = N(\epsilon)$. Now if we lower the value of m, it will only make $\|x_n - x_m\|$ smaller, but we cannot lower m below $N(\epsilon)$, otherwise our taking $n \to \infty$ would cause problems. So we have that for $n, m > \frac{1}{2\epsilon}$ then $\|x_n - x_m\| < \epsilon$. Hence the sequence is Cauchy.

b) Now imagine that the sequence was instead in a space which has as its norm $\|x\| = \max_{t\in[0,1]} |x(t)|$. In this case, prove that the sequence is not Cauchy.

For the second part, consider the "curves" that were given for this function in class. If you take any two of these with values n and m where $n < m$, the largest difference between the two curves happens when the curve associated with m starts to rise from zero. Look at the picture and think about this and it should be obvious. This "largest distance" is the norm defined in this case: $\|x_n - x_m\| = \max_{t\in[0,1]} |x_n(t) - x_m(t)| = \left|x_n\left(\frac{1}{2}-\frac{1}{m}\right) - x_m\left(\frac{1}{2}-\frac{1}{m}\right)\right| = \left|x_n\left(\frac{1}{2}-\frac{1}{m}\right)\right| = 1 + n\left(\frac{1}{2}-\frac{1}{m}\right) - \frac{n}{2} = 1 - \frac{n}{m}$

We need this to be $\|x_n - x_m\| = 1 - \frac{n}{m} < \epsilon \quad \Rightarrow \quad \frac{n}{m} > 1 - \epsilon$. But there is no number $N(\epsilon)$ beyond which the values always satisfy this. To see this consider $\epsilon = \frac{1}{2} \quad \Rightarrow \quad \frac{n}{m} > \frac{1}{2}$. But to have $\frac{n}{m} < \frac{1}{2}$ requires taking $2n < m$ which is always possible. Hence this sequence is not Cauchy.

5. Show that the sequence $g_n(x) = \frac{\cos(nx)}{\sqrt{n}}$ converges uniformly to $g(x) = 0$ for $x \in \mathbb{R}$. Does it converge pointwise?

We compute $|g_n(x) - g(x)| = \left|\frac{\cos(nx)}{\sqrt{n}}\right| = \frac{|\cos(nx)|}{\sqrt{n}} \le \frac{1}{\sqrt{n}}$. Therefore for a given ϵ, if $n > \frac{1}{\epsilon^2}$, then $|g_n(x) - g(x)| < \epsilon$ for all $x \in \mathbb{R}$.

Since uniform convergence implies pointwise convergence, then yes it also converges pointwise.

6. Show that the sequence $f_n(x) = x^n$ on $x \in [0,1]$, converges pointwise but not uniformly. Does it converge in the mean?

First of all, this sequence converges to the discontinuous function

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

If we choose ϵ, then for convergence we need $|f(x) - f_n(x)| < \epsilon$ for all $n > N$. If N depends only on ϵ, then the convergence is uniform, whereas if it depends on both ϵ and x then it is pointwise.

For $x = 1$ we find: $|f(x) - f_n(x)| = |1 - 1^n| = 0 < \epsilon$ which works out for any n.

But for $x < 1$ we find:

$|f(x) - f_n(x)| = |0 - x^n| = x^n < \epsilon$ or $\ln(x^n) < \ln(\epsilon) \;\Rightarrow\; n\ln(x) < \ln(\epsilon) \;\Rightarrow\; n > \frac{\ln(\epsilon)}{\ln(x)}$ since $\ln(x) < 0$ for $x < 0$, which depends on x in a manner that cannot be removed by choosing a single fixed value of $N(\epsilon)$. Therefore the convergence is pointwise, but not uniform.

To have mean convergence we need $\int_0^1 |f(x) - f_n(x)|^2 dx < \epsilon$ for all $n > N(\epsilon)$.

Well, $\int_0^1 |f(x) - f_n(x)|^2 dx = \int_0^1 |f(x) - x^n|^2 dx = \int_0^1 [f(x)^2 - 2f(x)x^n + x^{2n}]dx$

Now $f(x)^2 = f(x)$ (think about it) and $f(x)x^n = f(x)$ as well. But recall that $f(x)$ is only nonzero over the point at $x = 1$, and hence the integral of this "almost zero" function using the Lebesgue measure would give 0. That leaves $\int_0^1 |f(x) - f_n(x)|^2 dx = \int_0^1 x^{2n} dx = \frac{1}{2n+1} < \epsilon \;\Rightarrow\; n > N(\epsilon) = \frac{1}{2\epsilon} - \frac{1}{2}$, which is obviously achievable!

7. Consider the function $g(x) = x^2 - x$ over the interval $x \in [0,1]$.

a) Construct the first two polynomials using the Weierstrass construction from class, i.e. calculate $P_1(x)$ and $P_2(x)$.

$$P_n(x) = c_n \int_0^1 g(t)[1 - (t-x)^2]^n dt$$

$$P_1(x) = c_1 \int_0^1 (t^2 - t)[1 - (t-x)^2]dt$$

$$= \frac{3}{4}\int_0^1 (t^2 - t^4 - x^2t^2 + 2xt^3 - t + t^3 + x^2t - 2xt^2)dt$$

$$= \frac{3}{4}\left[\frac{1}{3} - \frac{1}{5} - \frac{x^2}{3} + \frac{x}{2} - \frac{1}{2} + \frac{1}{4} + \frac{x^2}{2} - \frac{2x}{3}\right]$$

$$= -\frac{7}{80} - \frac{1}{8}x + \frac{1}{8}x^2$$

$$P_2(x) = c_2 \int_0^1 (t^2 - t)[1 - (t-x)^2]^2 dt$$

$$= \frac{15}{16}\int_0^1 (t^2 - t)[(1 - 2x^2 + x^4) + (4x - 4x^3)t + (6x^2 - 2)t^2 - 4xt^3 + t^4]dt$$

$$= \frac{15}{16}\left[\frac{1}{6}x^4 - \frac{1}{3}x^3 - \frac{1}{30}x^2 + \frac{1}{5}x + \frac{19}{210}\right]$$

$$= \frac{5}{32}x^4 - \frac{5}{16}x^3 - \frac{5}{160}x^2 + \frac{3}{16}x + \frac{19}{224}$$

b) Construct the Taylor series for the function to fourth order?

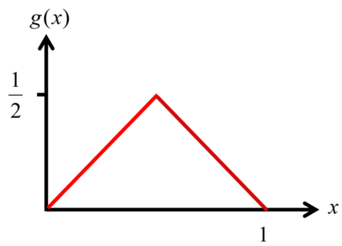
$$P_4'(x) = \frac{1}{0!}g(0) + \frac{1}{1!}\frac{dg}{dx}\Big|_0 x + \frac{1}{2!}\frac{d^2g}{dx^2}\Big|_0 x^2 + \frac{1}{3!}\frac{d^3g}{dx^3}\Big|_0 x^3 + \frac{1}{4!}\frac{d^4g}{dx^4}\Big|_0 x^4$$

$$= 0 - x + x^2 + 0x^3 + 0x^4 = -x + x^2$$

c) Which one wins? Taylor wins.

d) Consider instead the function given by $g(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ 1 - x & \frac{1}{2} \leq x \leq 1 \end{cases}$.

Which one wins in this case? Why?



Clearly Weierstrass wins. This is because Weierstrass is based on the integral of the continuous function over the entire interval. Taylor on the other is based on the existence of derivatives to all orders which this will not exhibit since its first derivative is discontinuous at $x = \frac{1}{2}$. If you think about it, doing the Taylor series around $x = 0$ will result in a straight line that goes on indefinitely. There is no way to encode the "kink" in the Taylor series, since Taylor only works from a single point. Weierstrass on the other hand works with the entire interval.